Exercise 1 –

1. Show that between two rational numbers there is always an irrational
2. Let $a, b \in \mathbb{Q}^+, a > 0, b > 0$ such that $\sqrt{a}$ and $\sqrt{b}$ are irrationals. Show that $\sqrt{a} + \sqrt{b}$ is irrational.
3. Show that there is no rational number solution of the equation : $x^3 = x + 7$.

Exercise 2 –

1. Show that $\forall x, y \in \mathbb{R}$ one has
$$\frac{|x+y|}{1+|x+y|} \leq \frac{|x|}{1+|x|} + \frac{|y|}{1+|y|}.$$
2. Let $x, y \in \mathbb{R}$. Show that
$$x = y \Leftrightarrow \forall \varepsilon > 0 : |x - y| < \varepsilon.$$

Exercise 3 – The symbol $[x]$ denotes the integer part of a real x. Prove that the following properties are true.

1. $\forall x \in \mathbb{R} : x - 1 < [x] \leq x.$
2. $\forall x \in \mathbb{R} - \mathbb{Z} : [-x] = -[x] - 1.$
3. $\forall x \in \mathbb{R} : \left[\frac{x}{2}\right] + \left[\frac{x+1}{2}\right] = [x].$
4. $\forall x, y \in \mathbb{R} : [x] + [y] \leq [x + y].$

Exercise 4 – Solve, in $\mathbb{R}$, the equation : $[2x + 1] = [x + 2]$.

Exercise 5 – Let A and B be nonempty bounded subsets of $\mathbb{R}$. Show that

1. $A \cup B$ is bounded,
2. $\sup(A \cup B) = \max(\sup A, \sup B)$,
3. $\inf(A \cup B) = \min(\inf A, \inf B)$.

Exercise 6 – Prove that the following sets are bounded. Determinate their $\sup, \inf, \max, \min$ if they exist

$$A = \left\{x \in \mathbb{R} : 0 \leq x \leq \sqrt{2}\right\}; B = \left\{\frac{1}{x}, 1 < x \leq 2\right\}$$

$$C = [-1, \sqrt{2}] \cap \mathbb{Q}, D = \left\{3 + \frac{1}{n}, n \in \mathbb{N}\right\}.$$

Exercise 7 – Write in the algebraic form the complex numbers

$$i^5 + i + 1; \frac{1+i}{1-i} - (1+2i)(2+2i) + \frac{3-i}{1+i};$$

$$(1+i)^3; \frac{2e^{i\frac{\pi}{4}}}{e^{i\frac{3\pi}{4}}}; \text{ the number of modulus 2 and argument } \frac{\pi}{3}.$$

Exercise 8 – Let $z = \sqrt{3} + i$. Compute

1. The square roots of z.
2. The cube roots of z.

Exercise 9 – Solve the equations :

— $\overline{z} = i(z - 1)$,
— $|z + 3i| = |z|$,
— $Re(z(1 + i)) + z\overline{z}$=0,
— $z^3 + 3z - 2i = 0$.

Exercise 10 – Use the Euler's formula

— to show that $\sin(a + b) = \sin a \cos b + \cos a \sin b; \forall a, b \in \mathbb{R}$;
— to linearize : $\sin^2 a \cos a, \sin^4 a; \forall a \in R$.

Supplementary exercises

Exercise 11 – Let x, y real numbers. Show that

$$[x] + [y] + [x + y] \leq [2x] + [2y].$$

Exercise 12 – **Bernoulli's inequality.** Prove that

$$(1 + \alpha)^n \geq 1 + n\alpha; \forall n \in \mathbb{N} \cup \{0\}, \forall \alpha \in]-1, +\infty[.$$

Exercise 13 – Consider the sets

$$A = \left\{\frac{2n+1}{n+3}, n \in \mathbb{N}\right\}; B = \left\{\frac{2n + (-1)^n}{n+1}, n \in \mathbb{N}\right\}$$

$$C = \left\{\frac{1-n}{1+n}, n \in \mathbb{N}\right\}.$$

Study the boundedness of these sets and determine whether their $\min, \max, \inf$ and $\sup$ exist.

Exercise 14 – Let A be a nonempty bounded subset of $\mathbb{R}$. Let us define

$$B = \{|x|; x \in A\}.$$

Show that

1. B is bounded,
2. $\sup B = \max\{|\inf A|, |\sup A|\}$,
3. $0 \leq \inf B \leq \min\{|\inf A|, |\sup A|\}$.

Exercise 15 –

1. Find $z \in \mathbb{C}$ such that $z^2 \in \mathbb{R}$.
2. Let the polynomial of complex variable
$$P(z) = z^3 - z^2 + z + 1 + a.$$
Find $a \in \mathbb{R}$ such that $z = -i$ is a root of P. Furthermore, for such value of a find factors of P.