Part I
Introduction to Optimization

1 Introduction to Optimization

Optimization is a branch of mathematics. In practice, one starts from a real-world problem, models it, and solves it mathematically either analytically (as an optimization problem) or numerically (as a mathematical program).

1.0.1 Framework

An optimization problem consists of, given a function $f : S \to \mathbb{R}$ finding:
1) Its minimum value v (or maximum) over the set S.
2)A point $x_0 \in S$ at which this minimum (or maximum) is attained, i.e., i.e. $f(x_0) = v$.

1.0.2 Vocabulary

– f f is the objective function
– v is the optimal value
– x_0 is the optimal solution
– $S = \{$the feasible set (the set of feasible solutions to the problem)$\}$
—- Problem formulation:: $\min_{x\in S} f(x)$resp.$\max_{x\in S} f(x)$
Remark: **Link Between Minimum and Maximum**
The Problem $\max_{x\in S} f(x)$ returns (x_0, v), then the problem $\min_{x\in S} f(x)$ returns $(x_0, -v)$.Hence the connection. Thus, the search for a maximum can always be reduced to the search for a minimum.

1.0.3 Applications

Optimization is used in many fields:
• In operations research (e.g., transportation problems, economics, inventory management...)
• In numerical analysis (e.g., approximation/solving of linear and nonlinear systems...)
• In control theory (e.g., system modeling, filtering...)

1.1 Different Types of Optimization

1.2 Classification of Optimization Problems

- Linear Optimization
f is a linear function : $f(x) =< c, x >$,
S is defined by affine functions : $ax + b \geq 0$.
-Quadratic Optimization

f is a convex quadratic function: $f(x) = \frac{1}{2} < Ax, x > + < b, x >$

A est une matrice symétrique semi-définie positive

S est défini par des fonctions affines : $ax + b \geq 0$,

-optimisation convexe

f is a convex function and S is a convex domain

– **Differentiable Optimization**

f is a differentiable function

S is defined by functions (=constraints) differentiable.

– Nondifferentiable Optimization

example: $f(x) = \max\{f_1(x), .., f_m(x)\}$.

-Infinite-dimensional Optimization

example: Variational problems $J(x) = \int_0^T L(t, x(t), x^{'}(t))dt$ where x belongs to a set of functions $X, x(0) = x_0$ et $x(T) = x_T$.

-Nonlinear Optimization

We distinguish three types of problems:

–Unconstrained problem: $\min_{x \in \mathbb{R}^n} f(x)$,

–Problem with equality constraints: $\min_{x \in S} f(x)$ with S of the form$S = \{x \in \mathbb{R}^n$

t.q. $g_i(x) = 0$ for $i = 1..l\}$and g_i: $\mathbb{R}^n \to \mathbb{R}$.

–Problem with inequality constraints:$\min_{x \in S} f(x)$ with S of the form $S = \{x \in \mathbb{R}^n$

such that . $h_i(x) \geq 0$ for $i = 1..l\}$ with h_i: $\mathbb{R}^n \to \mathbb{R}$.

1.3 Examples of Nonlinear Optimization Problems

A Production Problem

A factory needs n products $a_1, .., a_n$ in quantities $x_1, .., x_n$to manufacture a finished product b in quantity $h(x_1, .., x_n)$.*Let* p_i be the unit price of product i and p the selling price of the finished product.We want to calculate the optimal production. The goal is therefore to maximize the function $ph(x_1, .., x_n) - \sum_{i=1}^{i=n} x_i p_i$.

1.4 Existence of a solution

Weierstrass Theorem: Let f be continuous fonction on $S \subset \mathbb{R}^n$ with S closed and bounded set . Then $\exists\ x_0 \in S$, such that $f(x_0) = \min_{x \in S} f(x), \exists\ x_1 \in S$,such that $f(x_1) = \max_{x \in S} f(x)$.

This means that f is bounded on$S$ $(f(x_0) \leq f(x) \leq f(x_1))$ and attains its bounds. The maximum and minimum are attained in this setting.

Example

1) Let f : $\mathbb{R}$ in $\mathbb{R}$ is défined by $f(x) = x$. Let $S = [0, 1]$.

It is clear that the maximum is attained at 1and $\max_{x \in S} f(x) = f(1) = 1$.

S It is indeed a closed and bounded set. The conditions of the theorem are satisfied.

2) On the other hand, if we study the same function but this time on $S =]0, 1[$.

We also have $\max_{x\in S} f(x) = 1$. However, the maximum is not attained : $\nexists$ $x_0 \in S$ such that $f(x_0) = \max_{x\in S} f(x)$.

Here, the conditions of the theorem are not satisfied:S is not a closed and bounded set.

Remark: To ensure uniqueness, one must use properties specific to the objective function, such as strict convexity, for example.