Exercise 1 –

1. Binomial theorem. Show by induction that :
   $\forall x, y \in \mathbb{R}, \forall n \in \mathbb{N}$
$$(x+y)^n = \sum_{k=0}^{n} C_n^k x^{n-k} y^k \text{ where } C_n^k = \frac{n!}{k!(n-k)!}.$$
2. Show by induction that $\forall a \in \mathbb{R}, a \neq 1, \forall n \in \mathbb{N}$
$$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}.$$

Exercise 2 –

1. Using the definition of the limit, show that
$$\lim_{n\to+\infty} \frac{2n+1}{n+2} = 2, \lim_{n\to+\infty} \sqrt[n]{9} = 1, \lim_{n\to+\infty} \ln(1+n^2) = +\infty.$$
2. Calculate the limit of the following sequences
$$u_n = \frac{\frac{1}{2^n} - \frac{1}{5^n}}{2^n+1}, u_n = \sum_{k=1}^{n} \frac{1}{2^k}, u_n = \frac{n\sin(n)}{n^2+1},$$
$$u_n = n^2\left(\sqrt{3-\frac{2}{n}} - \sqrt{3}\right).$$

Exercise 3 – Suppose that (x_n) and (y_n) are convergent sequences of real numbers with the same limit l. Show that if (z_n) is a sequence such that
$$x_n \leq z_n \leq y_n, \forall n \in \mathbb{N}$$
then, (z_n) also converges to l.

Application. Prove that the $u_n = \sum_{k=1}^{n} \frac{\left[\frac{k}{2}\right]}{n^2}$ is a convergent sequence and find its limit.

Exercise 4 – Let (u_n) be the recurrent sequence
$$\begin{cases} u_1 = \frac{1}{2} \\ u_{n+1} = \frac{1}{2}(u_n^2+1), n \in \mathbb{N}. \end{cases}$$

1. Show by induction that for all $n \in \mathbb{N} : 0 < u_n < 1$.
2. Study the nature of the sequence.

Exercise 5 – Let a, b be real numbers such that $0 < a < b$. Define the sequences (u_n) and (v_n) by
$$\begin{cases} u_1 = a, v_1 = b \\ u_{n+1} = \sqrt{u_n.v_n} \\ v_{n+1} = \frac{1}{2}(u_n+v_n), n \in \mathbb{N}. \end{cases}$$
Prove that the sequences (u_n) and (v_n) are adjacent.

Exercise 6 –

— Show that $A_n = \sum_{k=1}^{n} \frac{1}{k^2}$ is a Cauchy sequence.

— Show that $B_n = \sum_{k=1}^{n} \frac{1}{k}$ is not a Cauchy sequence.

Exercise 7 – Show that the sequence $u_n = \sin\left(\frac{n\pi}{3}\right)$ has no limit. Hint : use the subsequences u_{3n} and u_{6n+1}.

Exercise 8 – Let (a_n) be the sequence
$$a_n = \frac{(-1)^n n}{n+5}, n \in \mathbb{N}.$$
Calculate the $\liminf_{n\to+\infty} a_n$ and $\limsup_{n\to+\infty} a_n$. Deduce an information on $Ad(a_n)$.

Supplementary exercises

Exercise 9 – Show that if (x_n) is a sequence that converges to zero then the sequence (x_n^2) also converges to zero.

Exercise 10 – Let (y_n) be the real sequence defined by
$$y_1 = \frac{1}{2}, y_{n+1} = y_n^2 + \frac{3}{16}, \forall n \in \mathbb{N}.$$

1. show that $\forall n \in \mathbb{N}$ we have $\frac{1}{4} < y_n < \frac{3}{4}$.
2. Is the sequence monotonic ? Deduce its nature.
3. Find $\inf\{y_n; n \in \mathbb{N}\}$ and $\sup\{y_n; n \in \mathbb{N}\}$.

Exercise 11 – Prove that the recurrent sequence defined by
$$u_1 = 1, u_{n+1} = \sqrt{u_n^2 + \frac{1}{2^n}}$$
is a Cauchy sequence.

Exercise 12 – Let a, b be positive real numbers. Let
$$u_1 = a, v_1 = b$$
and for all $n \in \mathbb{N}$
$$\frac{2}{u_{n+1}} = \frac{1}{u_n} + \frac{1}{v_n} \text{ and } v_{n+1} = \frac{1}{2}\left(u_n + v_n\right).$$
Show that the sequences (u_n) and (v_n) are adjacent.

Exercise 13 – Find the limit of the sequences
$$u_n = \left(1+\frac{1}{n}\right)^n, u_n = n\ln\left(1-\frac{1}{n^2}\right), u_n = \frac{n^{\sqrt{n+1}}}{(n+1)^{\sqrt{n}}}.$$

Exercise 14 – Find the $\liminf_{n\to+\infty} a_n$ and $\limsup_{n\to+\infty} a_n$ where
$$a_n = n^{\sin\left(\frac{n\pi}{2}\right)}, n \in \mathbb{N}.$$