Inequalities and Modes of Convergence

Ex 1

1. Let $X \sim \text{Exponential}(\lambda)$, $\lambda > 0$. Using Markov's inequality, find an upper bound for $P(X \geq a)$, where $a > 0$, and compare it with the exact value.

   - Using Chebyshev's inequality, find an upper bound for $P(|X - \mathbb{E}X| \geq b)$, $b > 0$.

2. Let $X \sim \text{Geometric}(p)$ with $\mathbb{P}(X = k) = (1-p)^{k-1}p$, $k = 1, 2, \ldots$

   Using Markov's inequality, find an upper bound for $P(X \geq a)$, where a is a positive integer, and compare with the exact value.

   Using Chebyshev's inequality, find an upper bound for $P(|X - \mathbb{E}X| \geq b)$.

3. The number of customers visiting a store during a day is modeled by a random variable X with mean $\mathbb{E}X = 100$ and variance $\text{Var}(X) = 225$. Using Chebyshev's inequality, find an upper bound for

$$P(X \leq 80 \text{ or } X \geq 120).$$

4. Let X be a positive random variable with $\mathbb{E}X = 10$. Discuss:

$$\mathbb{E}\left[\frac{1}{X+1}\right], \qquad \mathbb{E}\left[e^{1/(X+1)}\right], \qquad \mathbb{E}\big(\ln(X^{1/2})\big).$$

Ex 2 Let (X_n) be a sequence of random variables on the same probability space.

1. Let $X_n = \dfrac{1}{n}$. Show that $X_n \xrightarrow{\text{a.s.}} 0$.

2. Define

$$X_n = \begin{cases} 1, & \text{with probability } \dfrac{1}{n}, \\ 0, & \text{with probability } 1 - \dfrac{1}{n}. \end{cases}$$

   Show that we have convergence in probability but not almost surely.

Ex 3

Let X be a random variable with $\mathbb{E}[X] = 10$ and $\text{Var}(X) = 4$.

1. Use Chebyshev's inequality to find an upper bound for $\mathbb{P}(|X - 10| \geq 3)$.

2. Interpret the result.

3. Let $X_n \sim \text{Exponential}(n)$. Show that $X_n \xrightarrow{p} 0$.

4. Consider a sequence $\{X_n, n = 1, 2, 3, \dots\}$ such that

$$X_n = \begin{cases} -\dfrac{1}{n}, & \text{with probability } \dfrac{1}{2}, \\ \dfrac{1}{n}, & \text{with probability } \dfrac{1}{2}. \end{cases}$$

   Show that $X_n \xrightarrow{a.s.} 0$.

5. Let $X_1, X_2, X_3, \dots$ be independent random variables such that $X_n \sim \text{Bernoulli}\left(\dfrac{1}{n}\right)$ for $n = 2, 3, \dots$. Check whether $X_n \xrightarrow{a.s.} 0$.

6. Let $X_1, X_2, X_3, \dots$ be a sequence of random variables such that

$$X_n \sim \text{Geometric}\left(\frac{\lambda}{n}\right), \quad n = 1, 2, 3, \dots,$$

   where $\lambda > 0$ is a constant. Define a new sequence Y_n as

$$Y_n = \frac{1}{n} X_n, \quad n = 1, 2, 3, \dots$$

   Show that Y_n converges in distribution to Exponential(λ).

7. Let $X_1, X_2, X_3, \dots$ be i.i.d. random variables with distribution Uniform$(0, 1)$. Define the sequence

$$Y_n = \min(X_1, X_2, \dots, X_n).$$

   Prove the following convergence results independently (do not derive weaker convergence from stronger ones):

   (a) $Y_n \xrightarrow{d} 0$,

   (b) $Y_n \xrightarrow{p} 0$,

   (c) $Y_n \xrightarrow{L^r} 0$, for all $r \geq 1$,

   (d) $Y_n \xrightarrow{a.s.} 0$.

8. Let $X_1, X_2, X_3, \dots$ be a sequence of random variables such that

$$X_n \sim \text{Poisson}(n\lambda), \quad n = 1, 2, 3, \dots,$$

   where $\lambda > 0$ is a constant. Define

$$Y_n = \frac{1}{n} X_n, \quad n = 1, 2, 3, \dots$$

   Show that Y_n converges in mean square to λ, i.e. $Y_n \xrightarrow{m.s.} \lambda$.

9. Let $X_n \sim \mathcal{N}(0, 1/n)$. Show $X_n \xrightarrow{\mathcal{L}} 0$ (the degenerate law at 0) and that $X_n \xrightarrow{\mathbb{P}} 0$. Does $X_n \xrightarrow{a.s.} 0$?