TD4 — Limit Theorems

Exercise 1

1. Let $X_1, \dots, X_n$ be i.i.d. random variables with mean μ and variance σ^2. Show that

$$\frac{X_1^2 + \cdots + X_n^2}{n} \xrightarrow{\mathbb{P}} c.$$

Find the constant c.

2. Let $X_1, \dots, X_n$ be i.i.d. uniform on $[0, 1]$. Consider the geometric mean

$$G_n = (X_1 X_2 \cdots X_n)^{1/n}.$$

Show that G_n converges in probability to a constant C and find C.

3. Let $X_1, \dots, X_n$ be i.i.d. random variables with probability mass function

$$\mathbb{P}(X_1 = m) = \frac{(\log 2)^m}{2\,m!}, \qquad m = 0, 1, 2, \dots$$

Let $S_n = X_1 + \cdots + X_n$. Which of the following sequences converges to 0 in probability?

(a) $\dfrac{S_n}{n \log 2}$

(b) $\dfrac{S_n - n \log 2}{n}$

(c) $\dfrac{S_n - \log 2}{n}$

(d) $\dfrac{S_n - n}{\log 2}$

4. Let $X_1, \dots, X_n$ be i.i.d. with density $f(x) = e^{-x}$ for $x \geq 0$. Let $S_n = X_1 + \cdots + X_n$ and $\overline{X}_n = S_n / n$. For which of the following sequences does the distribution *not* converge to a normal distribution?

(a) $\dfrac{S_n - n}{\sqrt{n}}$

(b) $\dfrac{S_n}{\sqrt{n}}$

(c) $\sqrt{n}(\overline{X}_n - 1)$

(d) $\dfrac{\sqrt{n}(\overline{X}_n - 1)}{2}$

5. Let $X_1, \ldots, X_n$ be i.i.d. with mean 5 and variance 9. As $n \to \infty$,

$$\frac{X_1^2 + \cdots + X_n^2}{n} \xrightarrow{\mathbb{P}} \beta.$$

The value of β is: 14, 34, 28, 25.

6. Let $X_1, \ldots, X_n$ be i.i.d. with mean μ and variance σ^2. As $n \to \infty$,

$$\frac{X_1^2 + \cdots + X_n^2}{n} \xrightarrow{\mathbb{P}} \beta.$$

The value of β is: σ^2/n, $\mu^2 + \sigma^2$, $(\mu^2 + \sigma^2)/n$, μ^2.

7. Let $X_i \sim \mathrm{Exp}(1)$ i.i.d. Define $S_n = \sum_{i=1}^n X_i^2$. Then S_n/n converges in probability to: 0, 0.5, 1, 2.

Let $X_i \sim \mathcal{N}(0, 4)$ i.i.d. Define $S_n = \sum_{i=1}^n X_i^2/n$. Which of the following statements are correct?

1. $\sqrt{n}(S_n - 4)$ converges in distribution to $\mathcal{N}(0, 4)$.
2. S_n converges in distribution to 4.
3. $\lim_{n\to\infty} n \operatorname{Var}(S_n) = 32$.
4. S_n converges in distribution to $\mathcal{N}(0, 1)$.

Exercise 2

1. Toss a fair coin n times. Let $X_i = 1$ for heads and 0 for tails. Show that $\overline{X}_n \to 0.5$ in probability.
2. Let $X_1, \ldots, X_n$ be i.i.d. with distribution $\Gamma(4, 2)$. Let $S_n = \sum_{i=1}^n X_i$. Compute:

$$\lim_{n\to\infty} \mathbb{P}(S_n \leq 2n).$$

Options: (a) 0, (b) 0.5, (c) 1, (d) does not exist

Exercise 3

1. A coin is tossed 200 times. Approximate:

$$\mathbb{P}(80 \leq H \leq 120),$$

where H is the number of heads.

2. A random sample of size 100 is taken from a population with mean 60 and variance 400. Using the CLT, approximate:

$$\mathbb{P}\left(|\overline{X}_n - 60| \leq d\right)$$

for $d = 4$.

3. Let $X_1, X_2, \ldots, X_n$ be a sequence of i.i.d. random variables with $\mathbb{E}[X_1] = \mu$ and $\text{Var}(X_1) = \sigma^2 < \infty$.

   (a) Define $Y_i = X_i^2$. Determine the almost sure limit of

   $$A_n = \frac{1}{n}\sum_{i=1}^{n} Y_i \quad \text{as } n \to \infty.$$

   (b) Let $X_i \sim \text{Exp}(1)$. Define

   $$B_n = \frac{1}{n}\sum_{i=1}^{n} X_i^2.$$

   Find the almost sure limit of B_n.

Exercise 4

1. Let $X_i \sim \text{Bernoulli}(0.4)$, $n = 100$. Approximate:

$$\mathbb{P}\left(\sum_{i=1}^{100} X_i \geq 50\right).$$

2. In a communication system, each data packet contains 1000 bits. Each bit is received in error with probability 0.1, independently. Compute:

$$\mathbb{P}(\text{more than 120 errors}).$$

3. A random sample of size 100 is taken from a population with mean 60 and variance 400. Using the CLT, find the probability that the sample mean differs from μ by at most 4.

4. Let X_i be i.i.d. with mean 3 and variance 1/2. For $S_n = X_1 + \cdots + X_n$ with $n = 120$, estimate:

$$\mathbb{P}(340 < S_n < 370).$$