Question 1: What is a composition of two functions?

Many operations on functions are very familiar. Operations such as addition, subtraction, multiplication and division are similar to the same operations on real numbers. If we have two functions $f(x)$ and $g(x)$, applying these operations is straightforward:

$$(f+g)(x) = f(x) + g(x)$$

$$(f-g)(x) = f(x) - g(x)$$

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

In each case, the operation is performed by taking the output from each function and adding, subtracting, multiplying or dividing the resulting outputs.

For instance, if the functions are given by $f(x) = \sqrt{x+1}$ and $g(x) = \frac{1}{x}$, then

$$(f+g)(x) = \sqrt{x+1} + \frac{1}{x}$$

$$(f-g)(x) = \sqrt{x+1} - \frac{1}{x}$$

$$(f \cdot g)(x) = \sqrt{x+1} \cdot \frac{1}{x}$$

$$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x+1}}{\frac{1}{x}}$$

The output of each function, represented by its formula, is added, subtracted, multiplied, or divided to obtain the corresponding output. The resulting formulas may be simplified, but it is not necessary to simplify the resulting formula to perform the operation.

The composition of two functions works a bit differently.

The Composition of Functions

Let $f(x)$ and $g(x)$ be functions. The composition of f and g is a function whose outputs are given by $f\left(g(x)\right)$ for all values of x in the domain of $g(x)$ such that the output of $g(x)$ is in the domain of $f(x)$.

For the functions $f(x)=\sqrt{x+1}$ and $g(x)=\dfrac{1}{x}$, the composition of f and g (read as f of g of x) is

$$f\left(g(x)\right)=f\left(\frac{1}{x}\right)$$ Replace $g(x)$ with its output $\dfrac{1}{x}$

$$=\sqrt{\frac{1}{x}+1}$$ Replace x in $f(x)$ with $\dfrac{1}{x}$

If the order of the functions $f(x)=\sqrt{x+1}$ and $g(x)=\dfrac{1}{x}$ are reversed in the composition, the order in which we carry out the composition must also be reversed. The composition of g and f (read as g of f of x) is

$$g\left(f(x)\right)=g\left(\sqrt{x+1}\right)$$ Replace $f(x)$ with its output $\sqrt{x+1}$

$$=\frac{1}{\sqrt{x+1}}$$ Replace x in $g(x)$ with $\sqrt{x+1}$

Using these functions, we can substitute any reasonable number for x to find the value of the composition. For instance, the value of $g\left(f(3)\right)$ is

$$g\left(f(3)\right)=\frac{1}{\sqrt{3+1}}=\frac{1}{2}$$

However, $x=-1$ is not a reasonable substitution. Although we can find $f(-1)=\sqrt{-1+1}=0$, the output 0 is not in the domain of $g(x)=\frac{1}{x}$. Other values, such as $x=-2$, are also not reasonable since they are not in the domain of $f(x)=\sqrt{x+1}$.

Example 1 Find the Composition

Suppose you have two functions, $f(x)=x^3$ and $g(x)=x^2-2x$. Form each of the compositions below.

a. $f(g(x))$

Solution The composition $f(g(x))$ is formed by substituting the formula for $g(x)$ into the formula for $f(x)$:

$$f(g(x))=f(x^2-2x) \quad \text{Replace } g(x) \text{ with } x^2-2x$$

$$=(x^2-2x)^3 \quad \text{Substitute } x^2-2x \text{ into } f(x)$$

The composition is $f(g(x))=(x^2-2x)^3$.

b. $g(f(x))$

Solution The composition $g(f(x))$ is formed by substituting the formula $f(x)=x^3$ into the function $g(x)=x^2-2x$,

$$g(f(x))=g(x^3) \quad \text{Replace } f(x) \text{ with } x^3$$

$$=(x^3)^2-2(x^3) \quad \text{Substitute } x^3 \text{ into } g(x)$$

$$=x^6-2x^3 \quad \text{Simplify the exponents on the first term}$$

The composition is $g(f(x))=x^6-2x^3$.